Fuzzy Volterra Integral Equations with Piecewise Continuous Kernels: Theory and Numerical Solution

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Abstract

This study aims to discuss the existence and uniqueness of solution of fuzzy Volterra integral equations with piecewise continuous kernels. These types of problems are often encountered in balancing issues for systems with hereditary dynamics, such as electric load leveling. The method of successive approximations is applied and the main theorems are proved based on the method. Some examples are discussed and the results are presented for different values of μ by plotting several graphs.

keywords: Fuzzy Volterra integral equation; Piecewise kernel; Successive approximation; Error estimation.

1 Introduction

Fuzzy integral equations (FIEs) are among applicable and important problems of Engineering, Physics, Biology, Chemistry and many other fields. Bede and Gal [3], Friedman and Ma [5] and Goetschel and Voxman [6] have some studies on theory of FEIs. Ziari and Abbasbandy solved nonlinear FEIs using fuzzy quadrature rules [11]. The Reproducing Kernel Hilbert space method has been applied by Javan et al. in [12], the radial basis functions has discussed by Asari et al in [13]. Amirfakhrian et al. used the fuzzy interpolation techniques for solving FIEs in [14]. Also many other techniques for solving FIEs can be found in [15]. In [17] the well-known sinc-collcation method in both DE and SE precisions were used for solving fuzzy Fredholm integral equations. In [18] combining of the homotopy analysis method and Laplace transformations were applied to study the FEIs of the Abel type. In [19, 20] the CESTAC method and the CADNA library were employed to identify the optimal results of the homotopy analysis method for solving FIEs. Also the numerical solution of the fuzzy Volterra integral equation with weakly

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singular kernel and its smoothness of the solution have been discussed by Alijani and Kangro in [21, 22].

Volterra integral equation with piecewise continuous kernel is known and applicable problem which can be employed in various balance problems including electric loading problem. Sidorov et al. in [23, 24] studied the generalized solution of Volterra integral equations. Solvability of this problem has been illustrated by Sidorov in [26, 27] and Muftahov and Sidorov in [25]. The successive approximation method was used to find the solution of Volterra integral equations in [28]. The numerical solution of this problem can be found in [29]. Also some numerical and semi-analytical methods can be found for solving Volterra integral equations with piecewise kernel such as the spline collocation method [30], Lagrange-collocation method [31], Adomian decomposition method [33], homotopy perturbation method [34], the collocation method with Taylor polynomials [35] and other [32]. For more details on the theory of Volterra integral equations with piecewise continuous kernels readers may refer to monograph [1]. Such equations naturally generalizes the non-classic Volterra equations studied in monograph [2].

This study deals with the novel class of fuzzy Voltera integral equation (FVIE) with piecewise continuous kernel

$$Z(v) = Y(v) \oplus (\mathcal{FR}) \sum_{t=1}^{m'} \int_{\theta_{t-1}(v)}^{\theta_t(v)} K_t(r,v) \odot G(Z(r))dr, \quad z_1 \leq s, v \leq T \leq z_2, \tag{1}$$

where

$$z_1 =: \theta_0(v) < \theta_1(v) < ... < \theta_{m'-1}(v) < \theta_{m'}(v) := v, \quad z_1 \leq v \leq T \leq z_2$$

and the kernel $K_t(r,v)$ is a crisp and positive function over the square $z_1 \leq s, v \leq T \leq z_2$, $Z(v)$ shows a fuzzy real valued function and $G : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is continuous. Also $K_t(r,v)$ is a piecewise kernel along continuous curves $\theta_t(v), t = 1,2,...,m'$, therefore $K_1(r,v), K_2(r,v), ..., K_{m'}(r,v)$ are uniformly continuous with respect to t and there exist $M_t > 0$ such that $M_t = \max_{z_1 \leq r,v \leq z_2} |K_t(r,v)|$. We applied the successive approximations for solving problem (1). The existence of solution theorem is also discussed. Also the main theorem is proved below to show the error estimation of the problem. Solving some examples in both linear and nonlinear and plotting error graphs and also graphs of fuzzy approximate solutions, the ability and efficiency of the method are shown.

This paper is organized as follows. Section 2 provides the preliminaries of fuzzy mathematics. Section 3 is the main idea of this study. Also in this section the main existence of solution theorem is illustrated. Section 4 shows the error estimation of the successive approximation method for solving problem (1). Section 5 provides the linear and nonlinear examples. Using some graphs we show the accuracy of the method. Section 5 is the conclusion.

2 Preliminaries

We have reported the main definitions and theorems of fuzzy mathematics [3, 4, 5, 6, 7, 8].

Definition 1. *Based on the following properties a fuzzy number $p : \mathbb{R} \to [0,1]$ can be defined as a function:*

1. p *is normal which is* $\exists x_0 \in \mathbb{R}; p(x_0) = 1$,

2. p *is fuzzy convex set* $p(\gamma x + (1-\gamma)y) \geq \min\{p(x), p(y)\}, \forall x, y \in \mathbb{R}, \gamma \in [0,1]$.

3. p *is upper semi-continuous on* $\mathbb{R}$,

4. $\{x \in \mathbb{R} : p(x) > 0\}$ *is a compact set.*

$\mathbb{R}_F$ shows all fuzzy numbers sets.

Definition 2. $(\underline{p}(\mu), \overline{p}(\mu)), 0 \leq \mu \leq 1$ *is the parametric form of an arbitrary fuzzy number satisfying the following conditions:*

1. $\underline{p}(\mu)$ *is a bounded left continuous non-decreasing function over* $[0,1]$,

2. $\overline{p}(\mu)$ *is a bounded left continuous non-increasing function over* $[0,1]$,

3. $\underline{p}(\mu) \leq \overline{p}(\mu), 0 \leq \mu \leq 1$

We show the scalar multiplication and addition of fuzzy numbers as:

1. $(p \oplus p_1)(\mu) = (\underline{p}(\mu) + \underline{p_1}(\mu), \overline{p}(\mu) + \overline{p_1}(\mu))$,

2. $(\gamma \odot p)(\mu) = \begin{cases} (\gamma \underline{p}(\mu), \gamma \overline{p}(\mu)) & \gamma \geq 0, \\ (\gamma \overline{p}(\mu), \gamma \underline{p}(\mu)) & \gamma < 0. \end{cases}$

Definition 3. *Let* $p = (\underline{p}(\mu), \overline{p}(\mu)), p_1 = (\underline{p_1}(\mu), \overline{p_1}(\mu))$ *be two fuzzy numbers then the distance can be defined as*

$$\mathcal{D}(p, p_1) = \sup_{\mu \in [0,1]} \max\{|\underline{p}(\mu) - \underline{p}(\mu)|, |\overline{p}(\mu) - \overline{p_1}(\mu)|\}.$$

We have the following properties for distance $\mathcal{D}$.

Theorem 1.

1. $(\mathbb{R}_F, \mathcal{D})$ *is a complete metric space,*

2. $\mathcal{D}(p \oplus p_2, p_1 \oplus p_2) = \mathcal{D}(p, p_1) \forall p, p_1, p_2 \in \mathbb{R}_F$,

3. $\mathcal{D}(k \odot p, k \odot p_1) = |k| \mathcal{D}(p, p_1), \forall p, p_1 \in \mathbb{R}_F \forall k \in \mathbb{R}$,

4. $\mathcal{D}(p \oplus p_1, p_2 \oplus p_3) \leq \mathcal{D}(p, p_2) + \mathcal{D}(p_1, p_3) \forall p, p_1, p_2, p_3 \in \mathbb{R}_F$.

Theorem 2.

1. *We have a commutative semigroup for* $(\mathbb{R}_F, \oplus)$ *with the zero element* $(\mathbb{R}_F, \oplus)$.

2. *There is no opposite element if there are fuzzy numbers which are not crisp ((*$\mathbb{R}_F, \oplus$*) cannot be a group).*

3. $\forall z_1, z_2 \in \mathbb{R}$ *with* $z_1, z_2 \geq 0$ *or* $z_1, z_2 \leq 0$ *and* $\forall p \in \mathbb{R}_F$, *one get* $(z_1 + z_2) \odot p = z_1 \odot p \oplus z_2 \odot u$.

4. $\forall \gamma \in \mathbb{R}$ *and* $p, p_1 \in \mathbb{R}_F$, *one get* $\gamma \odot (p \oplus p_1) = \gamma \odot p \oplus \gamma \odot p_1$

5. $\forall \gamma, \mu \in \mathbb{R}$ *and* $p \in \mathbb{R}_F$, *one get* $\gamma \odot (\mu \odot p) = (\gamma\mu) \odot p$.

6. *There is the general attributes of the norm for of* $\|.\|_F : \mathbb{R}_F \to \mathbb{R}$ *by* $\|p\|_F = \mathcal{D}(p, \tilde{0})$ *which is* $\|p\|_F = 0 \Leftrightarrow p = \tilde{0}$, $\|\gamma \odot p\|_F = |\gamma| \|p\|_F$ *and* $\|p \oplus p_1\|_F \leq \|p\|_F + \|p_1\|_F$

7. $|\|p\|_F + \|p_1\|_F| \leq \mathcal{D}(p, p_1)$ *and* $\mathcal{D}(p, p_1) \leq \|p\|_F + \|p_1\|_F$ *for any* $p, p_1 \in \mathbb{R}_F$.

Definition 4. *Continuity of a fuzzy real number valued function* $Y : [z_1, z_2] \to \mathbb{R}_F$ *can be defined in* $x_0 \in [z_1, z_2]$ *as* $\forall \varepsilon > 0,\ \exists \rho > 0;\ \mathcal{D}(Y(x), Y(x_0)) < \varepsilon$, *whenever* $x \in [z_1, z_2]$ *and* $|x - x_0| < \rho$.

Definition 5. *Assume that* $Y : [z_1, z_2] \to \mathbb{R}_F$ *is a bounded mapping. The modulus of continuity* $\omega_{[z_1,z_2]}(Y, .) : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$ *is defined as*

$$\omega_{[z_1,z_2]}(Y, \rho) = \sup\{\mathcal{D}(Y(x), Y(y)) : x, y \in [z_1, z_2], |x - y| \leq \rho\}. \tag{2}$$

Also $\omega_{[z_1,z_2]}(Y, \rho)$ *is the uniform modulus of continuity of* Y *if* $Y \in C_F[z_1, z_2]$.

Theorem 3. *We have the following properties for the modulus of continuity:*

1. $\mathcal{D}(Y(x), Y(y)) \leq \omega_{[z_1,z_2]}(Y, |x - y|)$ *for any* $x, y \in [z_1, z_2]$.
2. $\omega_{[z_1,z_2]}(Y, \rho)$ *is increasing function of* ρ,
3. $\omega_{[z_1,z_2]}(Y, 0) = 0$,
4. $\omega_{[z_1,z_2]}(Y, \rho_1 + \rho_2) \leq \omega_{[z_1,z_2]}(Y, \rho_1) + \omega_{[z_1,z_2]}(Y, \rho_2),\ \ \rho_1, \rho_2 \geq 0$
5. $\omega_{[z_1,z_2]}(Y, n\rho) \leq n\omega_{[z_1,z_2]}(Y, \rho)$ *for any* $\rho \geq 0$ *and* $n \in N$,
6. $\omega_{[z_1,z_2]}(Y, \gamma\rho) \leq (\gamma + 1)\omega_{[z_1,z_2]}(Y, \rho),\ \ \rho, \gamma \geq 0$,
7. *For* $[z_3, z_4] \subseteq [z_1, z_2]$ *one get* $\omega_{[z_3,z_4]}(Y, \rho) \leq \omega_{[z_1,z_2]}(Y, \rho)$.

Definition 6. *Assume that* $Y : [z_1, z_2] \to \mathbb{R}_F$. Y *is a Riemann integrable of fuzzy type to* $I(Y) \in \mathbb{R}_F$ *if* $\forall \varepsilon > 0,\ \exists \rho > 0$; $\forall$ *division* $P = \{[p, p_1] : \xi\}$ *of* $[z_1, z_2]$ *with the norms* $\Delta(p) < \rho$, *it holds*

$$\mathcal{D}\left(\sum_{p}^{*} (p_1 - p) \odot Y(\xi), I(Y) \right) < \varepsilon; \tag{3}$$

where $\sum^*$ *shows the fuzzy summation. Then*

$$I(Y) = (\mathcal{FR}) \int_{z_1}^{z_2} Y(x)dx.$$

And for $Y \in C_F[z_1, z_2]$ *it follows*

$$\underline{(\mathcal{FR}) \int_{z_1}^{z_2} Y(t; r)dt} = \int_{z_1}^{z_2} \underline{Y}(t; r)dt,$$

$$\overline{(\mathcal{FR}) \int_{z_1}^{z_2} Y(t; r)dt} = \int_{z_1}^{z_2} \overline{Y}(t; r)dt$$

Lemma 1. *If $Y, V : [z_1, z_2] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ are fuzzy and continuous functions, then $Y : [z_1, z_2] \to \mathbb{R}_+$ by $F(x) = \mathcal{D}(Y(x), V(x))$ is continuous on $[z_1, z_2]$ and*

$$\mathcal{D}\left((\mathcal{FR})\int_{z_1}^{z_2} Y(x)dx, (\mathcal{FR})\int_{z_1}^{z_2} V(x)dx\right) \leq \int_{z_1}^{z_2} \mathcal{D}(Y(x), V(x))dx. \tag{4}$$

Theorem 4. *Assume that $Y : [z_1, z_2] \to \mathbb{R}_{\mathcal{F}}$ is a Henstock integrable and a bounded function. Then for $z_1 = x_0 < x_1 < ... < x_n = z_2$ and $\xi_i \in [x_{i-1}, x_i]$ it gives:*

$$\mathcal{D}\left((\mathcal{FH})\int_{z_1}^{z_2} Y(t)dt, \sum_{i=1}^{n} {}^*(x_i - x_{i-1}) \odot Y(\xi_i)\right) \leq \sum_{i=1}^{n}(x_i - x_{i-1})\omega_{[x_i, x_{i-1}]}(Y, x_i - x_{i-1})$$

Corollary 1. *Let $Y : [z_1, z_2] \to \mathbb{R}_{\mathcal{F}}$ be a bounded and Henstock integrable function. Then*

1. $\mathcal{D}\left((\mathcal{FH})\int_{z_1}^{z_2} Y(t)dt, (z_2 - z_1) \odot Y(\frac{z_1+z_2}{2})\right) \leq \frac{z_2-z_1}{2}\omega_{[z_1,z_2]}(Y, \frac{z_2-z_1}{2})$

2. $\mathcal{D}\left((\mathcal{FH})\int_{z_1}^{z_2} Y(t)dt, \frac{z_2-z_1}{2} \odot (Y(z_1) \oplus Y(z_2))\right) \leq \frac{z_2-z_1}{2}\omega_{[z_1,z_2]}(Y, \frac{z_2-z_1}{2})$

3. $\mathcal{D}\left((\mathcal{FH})\int_{z_1}^{z_2} Y(t)dt, \frac{z_2-z_1}{6} \odot (Y(z_1) \oplus 4 \odot Y(\frac{z_1+z_2}{2}) \oplus Y(z_2))\right) \leq 2(z_2 - z_1)\omega_{[z_1,z_2]}(Y, \frac{z_2-z_1}{6})$.

3 Main Idea

In this section let us discuss the existence and uniqueness of the solution of problem (1) based on the successive approximations. Assume $X = \{Y : [z_1, z_2] \to \mathbb{R}_{\mathcal{F}} : Y \textit{ is continuous}\}$ is the continuous functions space with fuzzy distance $\mathcal{D}^*(Y, V) = \sup_{z_1 \leq v \leq z_2} \mathcal{D}(Y(v), V(v))$. Let $A : X \to X$ be a nonlinear integral operator. Application of A for the problem (1) gives

$$AZ(v) = Y(v) \oplus (\mathcal{FR})\sum_{v=1}^{m'}\int_{\theta_{t-1}(v)}^{\theta_t(v)} K_t(r, v) \odot G(Z(r))dr, \quad \forall s, v \in [z_1, z_2], \forall F \in X.$$

Then

Theorem 5. *Assume that the kernels $K_1(r, v), K_2(r, v), ..., K_{m'}(r, v)$, $z_1 \leq s, v \leq T \leq z_2$ are positive and continuous. Let function $Y(v)$ be a fuzzy continuous of v, $z_1 \leq v \leq T \leq z_2$. Moreover*

$$\exists L > 0;\ \mathcal{D}(G(Z_1(p)), G(Z_2(p_1))) \leq L\mathcal{D}(Z_1(p), Z_2(p_1)), \quad \forall p, p_1 \in [z_1, z_2].$$

If $c = \sum_{t=1}^{m'} M_t L(\theta_t - \theta_{t-1}) < 1$ then there is a unique solution $F^* \in X$ for the FVIE (1) based on the following successive approximations method:

$$\begin{cases} Z_0(v) = Y(v), \\ \\ Z_m(v) = Y(v) \oplus (\mathcal{FR})\displaystyle\sum_{t=1}^{m'}\int_{\theta_{t-1}(v)}^{\theta_t(v)} K_t(r, v) \odot G(Z_{m-1}(r))dr, \quad z_1 \leq r, v \leq T \leq z_2, \quad m \geq 1, \end{cases} \tag{5}$$

which is convergent to F^*. Also

$$\mathcal{D}(F^*(v), Z_m(v)) \leq \frac{c^{m+1}}{L(1-c)} M_0, \quad \forall t \in [z_1, z_2], \quad m \geq 1 \tag{6}$$

is the error bound for $M_0 = \sup_{z_1 \leq v \leq z_2} \|G(Y(v))\|_F$.

Proof: We use the Banach fixed point principle to prove the theorem. Let us show $A : X \to X$ and also prove the uniformly continuity of the operator A. We know the continuity of Z on the compact set of $[z_1, z_2]$ thus that is uniformly continuous and it follows

$$\forall \varepsilon_1 > 0 \exists \rho_1 > 0; \mathcal{D}(Z(v_1), Z(v_2)) < \varepsilon_1 \textit{ whenever } |v_1 - v_2| < \rho_1, \forall v_1, v_2 \in [z_1, z_2].$$

Also $K_t, t = 1, 2, ..., m'$ is uniformly continuous. Therefore for $\varepsilon_t > 0$ there is an estimate

$$|K_t(r, v_1) - K_t(r, v_2)| < \varepsilon_t \textit{ whenever } |v_1 - v_2| < \rho_t, \ \forall v_1, v_2 \in [z_1, z_2].$$

Assume that $\rho = \min\{\rho_1, \rho_2, ..., \rho_{m'}\}$ and $v_1, v_2 \in [z_1, z_2]$ with $|v_1 - v_2| < \rho_t$. Applying Lemma 1 and Theorem 1 one can write:

$$\begin{aligned}
&\mathcal{D}(A(F)(v_1), A(F)(v_2)) \\
&\leq \mathcal{D}(Y(v_1), Y(v_2)) + \mathcal{D}((\mathcal{FR})\sum_{t=1}^{m'} \int_{\theta_{t-1}(v_1)}^{\theta_t(v_1)} K_t(r, v_1) \odot G(Z(r))dr, (\mathcal{FR})\sum_{t=1}^{m'} \int_{\theta_{t-1}(v_2)}^{\theta_t(v_2)} K_t(r, v_2) \odot G(Z(r))dr) \\
&\leq \varepsilon_1 + L\sum_{t=1}^{m'} |K_t(r, v_1) - K_t(r, v_2)| \mathcal{D}((\mathcal{FR})\int_{\theta_{t-1}(v_1)}^{\theta_t(v_1)} G(Z(r))dr, (\mathcal{FR})\int_{\theta_{t-1}(v_2)}^{\theta_t(v_2)} \tilde{0}dr) \\
&\leq \varepsilon_1 + \sum_{t=1}^{m'} \varepsilon_t(\theta_t(v_1) - \theta_{t-1}(v_1))M_0
\end{aligned}$$

where $M_0 = \sup_{z_1 \leq s \leq T \leq z_2} \|G(Y(r))\|_F$. By choosing $\varepsilon_1 = \frac{\varepsilon}{m'+1}$ and $\varepsilon_t = \frac{\varepsilon}{(m'+1)(\theta_t(v_1) - \theta_{t-1}(v_1))M_0}$ we find $\mathcal{D}(A(F)(v_1), A(F)(v_2)) < \varepsilon$.

Thus $A(F)$ is uniformly continuous for any $F \in X$, and so continuous on $[z_1, z_2]$, and hence $A(X) \subset X$. Now, it can be proved the contracting map of the operator A. For $Z_1, Z_2 \in X$ and

$t \in [z_1, z_2]$

$$\mathcal{D}(A(Z_1)(v), A(Z_2)(v))$$

$$\leq \mathcal{D}(Y(v), Y(v)) + \mathcal{D}((\mathcal{FR})\sum_{t=1}^{m'} \int_{\theta_{t-1}(v)}^{\theta_t(v)} K_t(r,v) \odot G(Z_1(r))dr, (\mathcal{FR})\sum_{t=1}^{m'} \int_{\theta_{t-1}(v)}^{\theta_t(v)} K_t(r,v) \odot G(Z_2(r))dr)$$

$$\leq \sum_{t=1}^{m'} \int_{\theta_{t-1}(v)}^{\theta_t(v)} \mathcal{D}(K_t(r,v) \odot G(Z_1(r)), (\mathcal{FR})K_t(r,v) \odot G(Z_2(r)))dr$$

$$\leq L \sum_{t=1}^{m'} M_t(\theta_t(v) - \theta_{t-1}(v))\mathcal{D}^*(Z_1, Z_2) = C\mathcal{D}^*(Z_1, Z_2),$$

thus, $\mathcal{D}(A(Z_1)(v), A(Z_2)(v)) \leq C\mathcal{D}^*(Z_1, Z_2)$. As $C < 1$ and A is a contraction on the Banach space $(X, \mathcal{D}^*)$. Thus based on the Banach fixed point principle there is unique solution F^* in X for Eq. (1) and

$$\mathcal{D}(F^*(v), Z_m(v)) \leq \mathcal{D}^*(F^*, Z_m) \leq \frac{C^m}{1-C}\mathcal{D}^*(Z_0, Z_1), \quad z_1 \leq v \leq T \leq z_2, \quad m \geq 1.$$

Also one can write

$$\mathcal{D}^*(Z_0, Z_1) = \sup_{z_1 \leq v \leq z_2} \mathcal{D}(Y(v), Y(v) + (\mathcal{FR})\sum_{t=1}^{m'} \int_{\theta_{t-1}(v)}^{\theta_t(v)} K_t(r,v) \odot G(Z(r))dr)$$

$$\leq \sup_{z_1 \leq v \leq z_2} \sum_{t=1}^{m'} \int_{\theta_{t-1}(v)}^{\theta_t(v)} \mathcal{D}(\tilde{0}, K_t(r,v) \odot G(Z_0(r)))dr$$

$$\leq \sum_{t=1}^{m'} M_t \int_{\theta_{t-1}(v)}^{\theta_t(v)} \sup_{z_1 \leq v \leq z_2} \mathcal{D}(\tilde{0}, G(Z_0(r)))dr$$

$$= \sum_{t=1}^{m'} M_t(\theta_t(v) - \theta_{t-1}(v))M_0 = \frac{C}{L}M_0.$$

Now it can be introducde the following numerical method to find the approximate solution of (1). As

$$z_1 = v_0 < v_1 < ... < v_{n-1} < v_n = z_2$$

where $v_i = a + ih$ and $h = \frac{b-a}{n}$ and one have the following iterative procedure as

$$\begin{cases} y_0(v) = Y(v), \\ y_m(v) = Y(v) \oplus \sum_{t=1}^{m'} \frac{h}{2} \odot \Big[K_t(v_0, v) \odot G(y_{m-1}(v_0)) \oplus K_t(v_n, v) \odot G(y_{m-1}(v_n)) \\ \qquad\qquad\qquad \oplus\, 2 \sum_{l=1}^{n-1} K_t(v_l, v) \odot G(y_{m-1}(v_l)) \Big], \quad m \geq 1. \end{cases}$$

Also the compact form of the relation is

$$\begin{cases} y_0(v) = Y(v), \\ y_m(v) = Y(v) \oplus \sum_{t=1}^{m'} \sum_{l=1}^{n-1} \frac{h}{2} \odot \Big[K_t(v_l, v) \odot G(y_{m-1}(v_l)) \oplus K_t(v_l, v) \odot G(y_{m-1}(v_l)) \Big], \quad m \geq 1. \end{cases} \tag{7}$$

4 Error Estimation

Theorem 6. *Assume that the nonlinear FVIE (1) with kernel $K_t(r, v)$ along continuous curves $\theta_t(v), t = 1, 2, ..., m'$ with positive sign on $[z_1, z_2] \times [z_1, z_2]$, G continuous on $\mathbb{R}_{\mathcal{F}}$ and Y continuous on $[z_1, z_2]$. Moreover there exists $L > 0$ such that*

$$\mathcal{D}(G(Z_1(p)), G(Z_2(p_1))) \leq L.\mathcal{D}(Z_1(p), Z_2(p_1)), \quad \forall p, p_1 \in [z_1, z_2].$$

For $C_t = M_t L(z_2 - z_1) < 1$ where $M_t = \max_{z_1 \leq r, v \leq T \leq z_2} |K_t(r, v)|$, then the successive scheme (7) converges to the unique solution of (1), F and the error estimation can be obtained as:

$$\mathcal{D}^*(F, y_m) \leq \sum_{t=1}^{m'} \frac{C_t}{2(1 - C_t)} \omega_{\theta_{t-1}, \theta_t}(Y, h) + \sum_{t=1}^{m'} \frac{C_t^{m+1} L_1}{L(1 - C_t)} + \sum_{t=1}^{m'} \frac{C_t^2 + 2C_t}{2LM_t(1 - C_t)} (L_1 \omega_s(K_t, h) + L_2 \omega_t(K_t, h))$$

where

$$\omega_s(K_t, h) = \sup_{z_1 \leq v \leq T \leq z_2} \{\sup |K_t(x, v) - K_t(y, v)| : |x - y| \leq h\}, \ t = 1, 2, ..., m',$$

and

$$\omega_t(K_t, h) = \sup_{z_1 \leq s \leq T \leq z_2} \{\sup |K_t(r, v_1) - K_t(r, v_2)| : |v_1 - v_2| \leq h\}, \ t = 1, 2, ..., m'.$$

Proof: We know

$$Z_1(v) = Y(v) \oplus (\mathcal{FR}) \sum_{t=1}^{m'} \int_{\theta_{t-1}(v)}^{\theta_t(v)} K_t(r, v) \odot G(Z_0(r)) dr, \quad z_1 \leq s, v \leq T \leq z_2,$$

then

$$\mathcal{D}(Z_1(v), y_1(v)) = \mathcal{D}(Y(v), Y(v))$$

$$+\mathcal{D}\bigg((\mathcal{FR})\sum_{t=1}^{m'}\int_{\theta_{t-1}(v)}^{\theta_t(v)} K_t(r,v)\odot G(Z_0(r))dr,$$

$$\sum_{t=1}^{m'}\sum_{l=0}^{n-1}\frac{h}{2}\odot\big[K_t(v_l,v)\odot G(Z_0(v_t))\oplus K_t(v_l,v)\odot G(Z_0(v_l))\big]\bigg)$$

$$=\mathcal{D}\bigg(\sum_{l=0}^{n-1}\sum_{t=1}^{m'}(\mathcal{FR})\int_{\theta_{t-1}(v_l)}^{\theta_t(v_l)} K_t(r,v)\odot G(Y(r))dr,$$

$$\sum_{t=1}^{m'}\sum_{l=0}^{n-1}\frac{h}{2}\odot\big[K_t(v_l,v)\odot G(Y(v_l))\oplus K_t(v_{l+1},v)\odot G(Y(v_{l+1}))\big]\bigg)$$

$$\leq\sum_{t=1}^{m'}\sum_{l=0}^{n-1}\mathcal{D}\bigg((\mathcal{FR})\int_{\theta_{t-1}(v_l)}^{\theta_t(v_{l+1})} K_t(r,v)\odot G(Y(r))dr,$$

$$\frac{h}{2}\odot\big[K_t(v_l,v)\odot G(Y(v_l))\oplus K_t(v_{l+1},v)\odot G(Y(v_{l+1}))\big]\bigg)$$

$$\leq\sum_{t=1}^{m'}\sum_{l=0}^{n-1}\mathcal{D}\bigg((\mathcal{FR})\int_{\theta_{t-1}(v_l)}^{\theta_t(v_{l+1})} K_t(r,v)\odot G(Y(r))dr,$$

$$\frac{h}{2}\odot\big[K_t(r,v)\odot G(Y(v_l))\oplus K_t(r,v)\odot G(Y(v_{l+1}))\big]\bigg)$$

$$+\sum_{t=1}^{m'}\sum_{l=0}^{n-1}\mathcal{D}\bigg(\frac{h}{2}\odot\big[K_t(r,v)\odot G(Y(v_l))\oplus K_t(r,v)\odot G(Y(v_{l+1}))\big],$$

$$\frac{h}{2}\odot\big[K_t(v_l,v)\odot G(Y(v_l))\oplus K_t(v_{l+1},v)\odot G(Y(v_{l+1}))\big]\bigg)$$

Applying the second part of the first corollary and regarding to Lemma 4 in [11] one have:

$$\begin{aligned}
\mathcal{D}(Z_1(v), y_1(v)) \quad &\leq \sum_{t=1}^{m'}\sum_{l=0}^{n-1} |K_t(r,v)|\mathcal{D}\Big((\mathcal{FR})\int_{\theta_{t-1}(v_l)}^{\theta_t(v_{l+1})} G(Y(r))dr, \frac{h}{2}\odot\big[G(Y(v_l))\oplus G(Y(v_{l+1}))\big]\Big)\\
&\quad + \sum_{t=1}^{m'}\sum_{l=0}^{n-1}\mathcal{D}\Big(\frac{h}{2}\odot K_t(r,v)\odot G(Y(v_l)), \frac{h}{2}\odot K_t(v_l,v)\odot G(Y(v_l))\Big)\\
&\quad + \sum_{t=1}^{m'}\sum_{l=0}^{n-1}\mathcal{D}\Big(\frac{h}{2}\odot K_t(r,v)\odot G(Y(v_l)), \frac{h}{2}\odot K_t(v_{l+1},v)\odot G(Y(v_{l+1}))\Big)\\
&\leq \frac{h}{2}\sum_{t=1}^{m'}\sum_{l=0}^{n-1}|K_t(r,v)|\omega_{[v_l,v_{l+1}]}(G(Y),\frac{h}{2})\\
&\quad + \frac{h}{2}\sum_{t=1}^{m'}\sum_{l=0}^{n-1}|K_t(r,v)-K_t(v_r,v)|\mathcal{D}(G(Y(v_l)),\tilde{0})\\
&\quad + \frac{h}{2}\sum_{t=1}^{m'}\sum_{l=0}^{n-1}|K_t(r,v)-K_t(v_{l+1},v)|\mathcal{D}(G(Y(v_{l+1})),\tilde{0})\\
&\leq \sum_{t=1}^{m'}\frac{M_t(\theta_t-\theta_{t-1})}{2}\omega_{[\theta_{t-1},\theta_t]}(G(Y),h) + \sum_{t=1}^{m'}(\theta_t-\theta_{t-1})M_0\omega_s(K_t,h)\\
&= \sum_{t=1}^{m'}\frac{M_t(\theta_t-\theta_{t-1})}{2}\sup_{\theta_{t-1}\leq p,p_1\leq\theta_t}\{\mathcal{D}(G(Y(p)),G(Y(p_1))) : |u-p_1|\leq h\}\\
&\quad + \sum_{t=1}^{m'}(\theta_t-\theta_{t-1})M_0\omega_s(K_t,h)\\
&\leq \sum_{t=1}^{m'}\frac{M_t(\theta_t-\theta_{t-1})}{2}\sup_{\theta_{t-1}\leq p,p_1\leq\theta_t}\{L.\mathcal{D}(Y(p),Y(p_1)) : |u-p_1|\leq h\}\\
&\quad + \sum_{t=1}^{m'}(\theta_t-\theta_{t-1})M_0\omega_s(K_t,h)\\
&\leq \sum_{t=1}^{m'}\frac{M_tL(\theta_t-\theta_{t-1})}{2}\omega_{[\theta_{t-1},\theta_t]}(Y,h) + \sum_{t=1}^{m'}(\theta_t-\theta_{t-1})M_0\omega_s(K_t,h)
\end{aligned}$$

where $M_0 = \sup_{\theta_{t-1} \leq s \leq \theta_t} \|G(Y(r))\|_{\mathcal{F}}$ and $\omega_s(K_t, h)$ is the partial modules of continuity. Thus

$$\mathcal{D}(Z_1(v), y_1(v)) \leq \sum_{t=1}^{m'} \frac{C_t}{2} \omega_{[\theta_{t-1}, \theta_t]}(Y, h) + \sum_{t=1}^{m'} \frac{C_t}{LM_t} M_0 \omega_s(K_t, h).$$

We have

$$Z_2(v) = Y(v) \oplus (\mathcal{FR}) \sum_{t=1}^{m'} \int_{\theta_{t-1}(v)}^{\theta_t(v)} K_t(r, v) \odot G(Z_1(r)) dr,$$

therefore

$$\mathcal{D}(Z_2(v), y_2(v)) = \mathcal{D}(Y(v), Y(v))$$

$$+\mathcal{D}\bigg((\mathcal{FR})\textstyle\sum_{t=1}^{m'}\int_{\theta_{t-1}(v)}^{\theta_t(v)} K_t(r,v)\odot G(Z_1(r))dr,$$

$$\sum_{t=1}^{m'}\sum_{l=0}^{n-1}\frac{h}{2}\odot\big[K_t(v_l,v)\odot G(y_1(v_l))\oplus K_t(v_{l+1},v)\odot G(y_1(v_{l+1}))\big]\bigg)$$

$$\leq \mathcal{D}\bigg((\mathcal{FR})\sum_{t=1}^{m'}\int_{\theta_{t-1}(v)}^{\theta_t(v)} K_t(r,v)\odot G(Z_1(r))dr,$$

$$\sum_{t=1}^{m'}\sum_{l=0}^{n-1}\frac{h}{2}\odot\big[K_t(r,v)\odot G(Z_1(v_l))\oplus K_t(r,v)\odot G(Z_1(v_{l+1}))\big]\bigg)$$

$$+\mathcal{D}\bigg(\sum_{t=1}^{m'}\sum_{l=0}^{n-1}\frac{h}{2}\odot\big[K_t(r,v)\odot G(Z_1(v_l))\oplus K_t(r,v)\odot G(Z_1(v_{l+1}))\big],$$

$$\sum_{t=1}^{m'}\sum_{l=0}^{n-1}\frac{h}{2}\odot\big[K_t(r,v)\odot G(y_1(v_l))\oplus K_t(r,v)\odot G(y_1(v_{l+1}))\big]\bigg)$$

$$+\mathcal{D}\bigg(\sum_{t=1}^{m'}\sum_{l=0}^{n-1}\frac{h}{2}\odot\big[K_t(r,v)\odot G(y_1(v_l))\oplus K_t(r,v)\odot G(y_1(v_{l+1}))\big],$$

$$\sum_{t=1}^{m'}\sum_{l=0}^{n-1}\frac{h}{2}\odot\big[K_t(v_l,v)\odot G(y_1(v_l))\oplus K_t(v_{l+1},v)\odot G(y_1(v_{l+1}))\big]\bigg)$$

$$\leq \sum_{t=1}^{m'}\frac{M_t(\theta_t-\theta_{t-1})}{2}\omega_{[\theta_{t-1},\theta_t]}(G(Z_1),\frac{h}{2}) + \sum_{t=1}^{m'}\frac{M_t(\theta_t-\theta_{t-1})}{2}\Big[\mathcal{D}(G(Z_1(v_l)),G(y_1(v_l)))$$

$$+\mathcal{D}(G(Z_1(v_{l+1})),G(y_1(v_{l+1})))\Big] + \sum_{t=1}^{m'}(\theta_t-\theta_{t-1})M_1\omega_s(K_t,h)$$

$$\leq \sum_{t=1}^{m'}\frac{C_t}{2}\omega_{[\theta_{t-1},\theta_t]}(Z_1,h) + \sum_{t=1}^{m'}\frac{C_t}{2}\Big[\mathcal{D}(Z_1(v_l),y_1(v_l)) + \mathcal{D}(Z_1(v_{l+1}),y_1(v_{l+1}))\Big] + \sum_{t=1}^{m'}\frac{C_t}{LM_t}M_1\omega_s(K_t,h)$$

where $M_1 = \sup_{\theta_{t-1}\leq s\leq\theta_t}\|G(y_1(r))\|_{\mathcal{F}}$. Applying (5) and (7) and using induction for $m \geq 3$ it can be concluded

$$\begin{aligned}\mathcal{D}(Z_m(v), y_m(v)) \quad &\leq \sum_{t=1}^{m'} \frac{C_t}{2}\omega_{[\theta_{t-1},\theta_t]}(Z_{m-1}, h) + \sum_{t=1}^{m'} \frac{C_t}{2}\Big[\mathcal{D}(Z_{m-1}(v_l), y_{m-1}(v_l)) \\ &+\mathcal{D}(Z_{m-1}(v_{l+1}), y_{m-1}(v_{l+1}))\Big] + \sum_{t=1}^{m'} \frac{C_t}{LM_t} M_{m-1}\omega_s(K_t, h)\end{aligned} \tag{8}$$

where $M_{m-1} = \sup_{\theta_{t-1}\leq v\leq\theta_t} \|G(y_{m-1}(v))\|_F$. Taking supremum for $z_1 \leq v \leq z_2$ from (8) we have

$$\begin{aligned}&\mathcal{D}^*(Z_m, y_m) \leq \sum_{t=1}^{m'} \frac{C_t}{2}\omega_{[\theta_{t-1},\theta_t]}(Z_{m-1}, h) + \sum_{t=1}^{m'} C_t\mathcal{D}^*(Z_{m-1}, y_{m-1}) + \sum_{t=1}^{m'} \frac{C_t}{LM_t} M_{m-1}\omega_s(K_t, h), \\ &\mathcal{D}^*(Z_{m-1}, y_{m-1}) \leq \sum_{t=1}^{m'} \frac{C_t}{2}\omega_{[\theta_{t-1},\theta_t]}(Z_{m-2}, h) + \sum_{t=1}^{m'} C_t\mathcal{D}^*(Z_{m-2}, y_{m-2}) + \sum_{t=1}^{m'} \frac{C_t}{LM_t} M_{m-2}\omega_s(K_t, h), \\ &\vdots \\ &\mathcal{D}^*(Z_1, y_1) \leq \sum_{t=1}^{m'} \frac{C_t}{2}\omega_{[\theta_{t-1},\theta_t]}(Y, h) + \sum_{t=1}^{m'} \frac{C_t}{LM_t} M_0\omega_s(K_t, h).\end{aligned} \tag{9}$$

If one multiple the above inequality to $1, C_t, ..., C_t^{m-1}$ and find the summation then

$$\begin{aligned}\mathcal{D}^*(Z_m, y_m) \quad &\leq \sum_{t=1}^{m'} \frac{C_t}{2}\Big(\omega_{[\theta_{t-1},\theta_t]}(Z_{m-1}, h) + C_t\omega_{[\theta_{t-1},\theta_t]}(Z_{m-2}, h) + ... + C_t^{m-1}\omega_{[\theta_{t-1},\theta_t]}(Y, h)\Big) \\ &+\sum_{t=1}^{m'} \frac{C_t}{LM_t}\omega_s(K_t, h)(M_{m-1} + C_tM_{m-2} + ... + C_t^{m-1}M_0).\end{aligned} \tag{10}$$

Moreover for $v_1, v_2 \in [z_1, z_2]$ with $|v_1 - v_2 \leq h|$ one can write

$$\begin{aligned}
&\mathcal{D}(Z_m(v_1), Z_m(v_2)) \\
&= \mathcal{D}\Bigg(Y(v_1) \oplus \odot(\mathcal{FR}) \sum_{t=1}^{m'} \int_{\theta_{t-1}(v_1)}^{\theta_t(v_1)} K_t(r, v_1) \odot G(Z_{m-1}(r))dr, \\
&\qquad Y(v_2) \oplus \odot(\mathcal{FR}) \sum_{t=1}^{m'} \int_{\theta_{t-1}(v_2)}^{\theta_t(v_2)} K_t(r, v_2) \odot G(Z_{m-1}(r))dr\Bigg) \\
&\leq \mathcal{D}(Y(v_1), Y(v_2)) + \sum_{t=1}^{m'} \int_{\theta_{t-1}(v_1)}^{\theta_t(v_1)} |K_t(r, v_1) - K_t(r, v_2)| \mathcal{D}(G(Z_{m-1}(r)), \tilde{0})dr \\
&\leq \mathcal{D}(Y(v_1), Y(v_2)) + \sum_{t=1}^{m'} \frac{C_t}{LM_t} \omega_t(K_t, h) M'_{m-1},
\end{aligned}$$

where $\omega_t(K_t, h)$ is the partial modulus of continuity with respect to t. Let $M'_{m-1} = \sup_{\theta_{t-1} \leq s \leq \theta_t} \|G(Z_{m-1}(r))\|_{\mathcal{F}}$ then we can find the relation Z_m and Y as:

$$\omega_{[\theta_{t-1}, \theta_t]}(Z_m, h) \leq \omega_{[\theta_{t-1}, \theta_t]}(Y, h) + \sum_{t=1}^{m'} \frac{C_t}{LM_t} \omega_t(K_t, h) M'_{m-1}. \tag{11}$$

And if we substitute above inequality into (10) we obtain

$$\begin{aligned}
\mathcal{D}^*(Z_m, y_m) \quad &\leq \sum_{t=1}^{m'} \frac{C_t}{2}(1 + C_t + C_t^2 + ... + C_t^{m-1}) \omega_{[\theta_{t-1}, \theta_t]}(Y, h) \\
&+ \sum_{t=1}^{m'} \frac{C_t}{2LM_t} \omega_t(K_t, h)(C_t M'_{m-2} + C_t^2 M'_{m-3} + ... + C_t^{m-1} M'_0) \qquad (12) \\
&+ \sum_{t=1}^{m'} \frac{C_t}{LM} \omega_s(K_t, h)(M_{m-1} + C_t M_{m-2} + ... + C_t^{m-1} M_0).
\end{aligned}$$

Let $L_1 = \max_{0 \le i \le m-1}\{M_i\}$ and $L_2 = \max_{0 \le i \le m-2}\{M_i'\}$ thus

$$\begin{aligned}
\mathcal{D}^*(Z_m, y_m) \quad &\le \sum_{t=1}^{m'} \frac{C_t}{2}(\frac{1-C_t^m}{1-C_t})\omega_{[\theta_{t-1},\theta_t]}(Y,h) \\
&+ \sum_{t=1}^{m'} \frac{C_t}{2LM_t}\omega_t(K_t,h)(C_t + C_t^2 + ... + C_t^{m-1})L_2 \\
&+ \sum_{t=1}^{m'} \frac{C_t}{LM}\omega_s(K_t,h)(1 + C_t + ... + C_t^{m-1})L_1.
\end{aligned} \tag{13}$$

From other hand $\frac{1-C_t^m}{1-C_t} \le \frac{1}{1-C_t}, t = 1, 2, ..., m'$ for each $m \in N$ thus

$$\begin{aligned}
\mathcal{D}^*(Z_m, y_m) \quad &\le \sum_{t=1}^{m'} (\frac{C_t}{2(1-C_t)})\omega_{[\theta_{t-1},\theta_t]}(Y,h) \\
&+ \sum_{t=1}^{m'} \frac{C_t^2 + 2C_t}{2LM_t(1-C_t)}(L_1\omega_s(K_t,h) + L_2\omega_t(K_t,h)).
\end{aligned}$$

Applying the inequality (6) we can write

$$\begin{aligned}
\mathcal{D}^*(F, y_m) \quad &\le \mathcal{D}^*(F, Z_m) + \mathcal{D}^*(Z_m, y_m) \\
&\le \sum_{t=1}^{m'} (\frac{C_t^m}{1-C_t})\mathcal{D}^*(Z_1, Z_0) + \sum_{t=1}^{m'} (\frac{C_t}{2(1-C_t)})\omega_{[\theta_{t-1},\theta_t]}(Y,h) \\
&+ \sum_{t=1}^{m'} \frac{C_t^2 + 2C_t}{2LM_t(1-C_t)}(L_1\omega_s(K_t,h) + L_2\omega_t(K_t,h)).
\end{aligned}$$

Since

$$\begin{aligned}
\mathcal{D}(Z_1(v), Z_0(v)) \quad &= \mathcal{D}\left(Y(v) \oplus (\mathcal{FR})\sum_{t=1}^{m'} \int_{\theta_{t-1}(v)}^{\theta_t(v)} K_t(r,v) \odot G(Z_0(r))dr, Z_0(v)\right) \\
&\le \mathcal{D}\left((\mathcal{FR})\sum_{t=1}^{m'} \int_{\theta_{t-1}(v)}^{\theta_t(v)} K_t(r,v) \odot G(Z_0(r))dr, \tilde{0}\right)
\end{aligned}$$

we obtain

$$\begin{aligned}\mathcal{D}^*(Z_1,Z_0) &\leq \sum_{t=1}^{m'} M_t(\theta_t-\theta_{t-1}) \sup_{\theta_{t-1}\leq s\leq\theta_t} \mathcal{D}(G(Y(r)),\tilde{0})\\ &= \sum_{t=1}^{m'} \frac{C_t}{L} M_0 \leq \sum_{t=1}^{m'} \frac{C_t}{L} L_1.\end{aligned}$$

Thus we get

$$\begin{aligned}\mathcal{D}^*(F,y_m) &\leq \sum_{t=1}^{m'} (\frac{C_t}{2(1-C_t)})\omega_{[\theta_{t-1},\theta_t]}(Y,h) + \sum_{t=1}^{m'} (\frac{C_t^{m+1}L_1}{L(1-C_t)})\\ &+ \sum_{t=1}^{m'} \frac{C_t^2+2C_t}{2LM_t(1-C_t)}(L_1\omega_s(K_t,h)+L_2\omega_t(K_t,h)).\Box\end{aligned}$$

Remark 1. *As we know $C_t<1, t=1,2,...,m'$ and it shows $\lim_{m\to\infty} C_t^{m+1}=0, t=1,2,...,m'$. And we have*

$$\lim_{h\to 0}\omega_{[\theta_{t-1},\theta_t]}(Y,h)=0, \quad \lim_{h\to 0}\omega_s(K_t,h)=0, \quad \lim_{h\to 0}\omega_t(K_t,h)=0, \quad t=1,2,...,m'.$$

The convergence of this scheme can be obtained by $\lim_{m\to\infty,h\to 0}\mathcal{D}^*(F,y_m)=0$.

5 Numerical Results

In this section some examples are presented. We apply the mentioned method for solving the problems.This is the first time that the problem (1) has been solved and there are no other methods to compare with. But in order to show the accuracy of the method we compare to the homotopy analysis method. All the mentioned examples are simulated problems.

Example 1. *We consider the problem (1) with $K_1(r,v)=1+v-r, K_2(r,v)=v-1, m'=2$, $a=\theta_0(v)=0, \theta_1(v)=\frac{v}{3}$ and $\theta_2(v)=v$ where*

$$\underline{Y}=(-2+\mu)(-1+v^2)-\frac{2}{81}(-2+\mu)(-1+v)v(-27+13v^2)-\frac{1}{324}(-2+\mu)v(-108-90v+4v^2+3v^3),$$

$$\overline{Y}=(-2+\mu)(1+v^2)-\frac{2}{81}(-2+\mu)(-1+v)v(27+13v^2)-\frac{1}{324}(-2+\mu)v(108+90v+4v^2+3v^3),$$

and the exact solution $(\underline{F}(v),\overline{F}(v))=((\mu-2)(v^2-1),(\mu-2)(v^2+1))$. The comparison between the exact solution $(\underline{F}(v),\overline{F}(v))$ and approximate solution $(\underline{F}_{10}(v),\overline{F}_{10}(v))$ for $\mu=0.5$ can be found in Fig. 1. The absolute errors for $m=10$ are presented in Fig. 2. Also the graph of obtained solutions for various r is presented in Fig. 3. Table 1 is to show the comparison between the absolute errors of the successive approximation and the HAM.

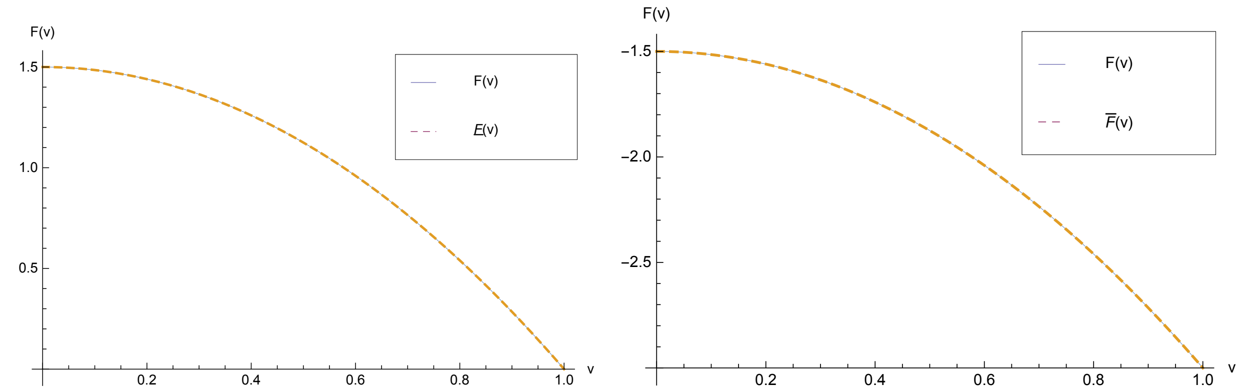

Figure 1: Comparison between the exact solution $(\underline{F}(v), \overline{F}(v))$ and approximate solution $(\underline{F}_{10}(v), \overline{F}_{10}(v))$ for $\mu = 0.5$.

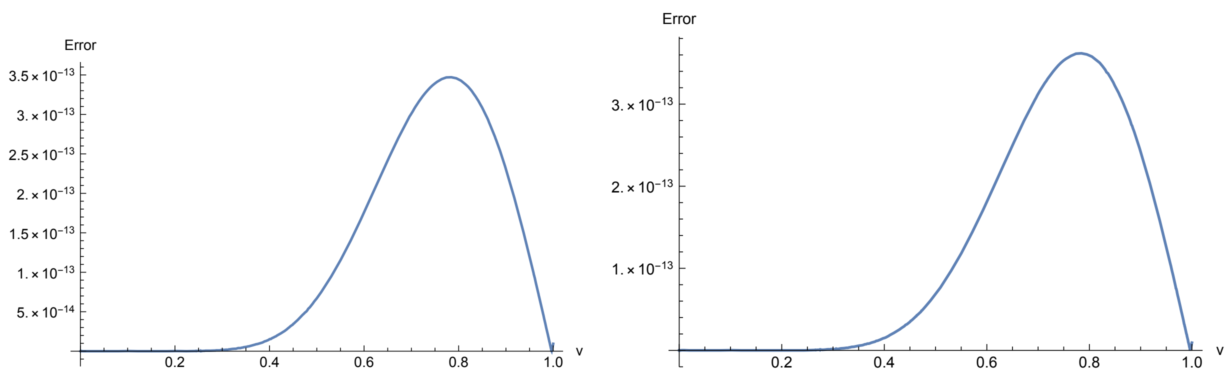

Figure 2: The absolute error for $(\underline{F}_{10}(v), \overline{F}_{10}(v))$ and $\mu = 0.5$.

Table 1: The errors of the successive approximation method and the HAM.

v	$\lvert\underline{F}(v) - \underline{F}_{10}(v)\rvert$	$\lvert\overline{F}(v) - \overline{F}_{10}(v)\rvert$	$\lvert\underline{F}(v) - \underline{F}_{HAM}(v)\rvert$	$\lvert\overline{F}(v) - \overline{F}_{HAM}(v)\rvert$
0.00	0	0	0	0
0.25	2.22045×10^{-16}	4.44089×10^{-16}	1.53211×10^{-14}	1.55431×10^{-14}
0.50	6.70575×10^{-14}	6.83897×10^{-14}	1.88627×10^{-12}	1.93756×10^{-12}
0.75	3.39395×10^{-13}	3.53051×10^{-13}	7.3529×10^{-12}	7.74225×10^{-12}
1.00	8.99281×10^{-15}	9.32587×10^{-15}	3.61256×10^{-13}	3.65485×10^{-13}

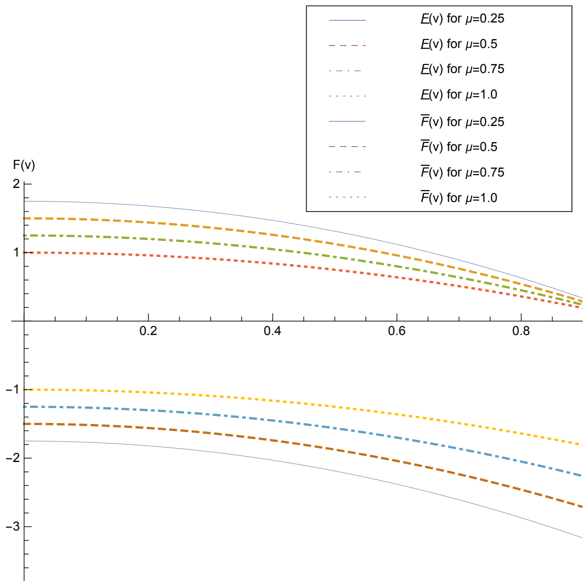

Figure 3: Fuzzy approximate solution for various μ.

Example 2. *We have* $K_1(r,v)=v, K_2(r,v)=v-1, K_3(r,v)=r-v, m'=3,\ z_1=\theta_0(v)=0, \theta_1(v)=\frac{v}{8}, \theta_2(v)=\frac{2v}{8}$ *and* $\theta_3(v)=v$, *with nonlinear term* $G(Z(r))=F^3(r)$ *where*

$$\underline{Y}=(1-\mu)v^3+\frac{(1023(-1+\mu)^3(-1+v)v^{10})}{10737418240}-\frac{(1073733109(-1+\mu)^3v^{11})}{118111600640},$$

$$\overline{Y}=(2+\mu)v^3-\frac{1023(2+\mu)^3(-1+v)v^{10}}{10737418240}+\frac{1073733109(2+\mu)^3v^{11}}{118111600640},$$

and the exact solution $(\underline{F}(v),\overline{F}(v))=((1-\mu)v^3,(\mu+2)v^3)$. *The comparative graphs between the exact and approximate solutions* $(\underline{F}_{20}(v),\overline{F}_{20}(v))$ *have been presented in Fig. 4 for* $\mu=0.5$. *Fig. 5 shows the absolute errors for both underline and overline cases. Fig. 6 demonstrates the approximate solutions* $(\underline{F}_{20}(v),\overline{F}_{20}(v))$ *for various* μ. *In order to show efficiency and accuracy of the method, we have compared the successive approximation method with the traditional homotopy analysis method. The results have been shown in Table 2.*

Table 2: The errors of the successive approximation method and the HAM.

v	$\lvert\underline{F}(v)-\underline{F}_{20}(v)\rvert$	$\lvert\overline{F}(v)-\overline{F}_{20}(v)\rvert$	$\lvert\underline{F}(v)-\underline{F}_{HAM}(v)\rvert$	$\lvert\overline{F}(v)-\overline{F}_{HAM}(v)\rvert$
0.00	0	0	0	0
0.25	0.0000253047	0.000126491	0.0000521732	0.000358749
0.50	0.00077713	0.00381917	0.00084267	0.00417518
0.75	0.00574238	0.0229733	0.0079563421	0.05269427
1.00	0.0228692	0.0211465	0.024871277	0.02565382

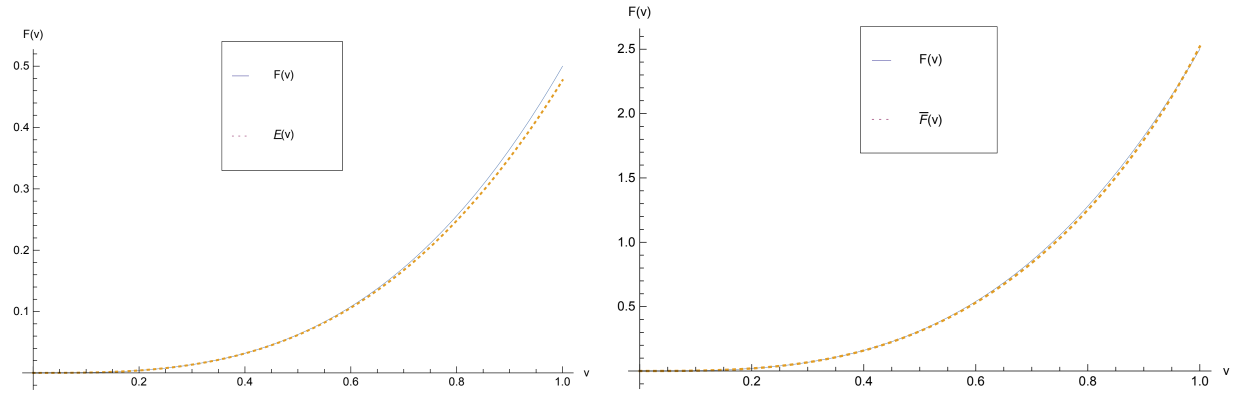

Figure 4: Comparison between the exact solution $(\underline{F}(v), \overline{F}(v))$ and approximate solution $(\underline{F}_{20}(v), \overline{F}_{20}(v))$ for $\mu = 0.5$.

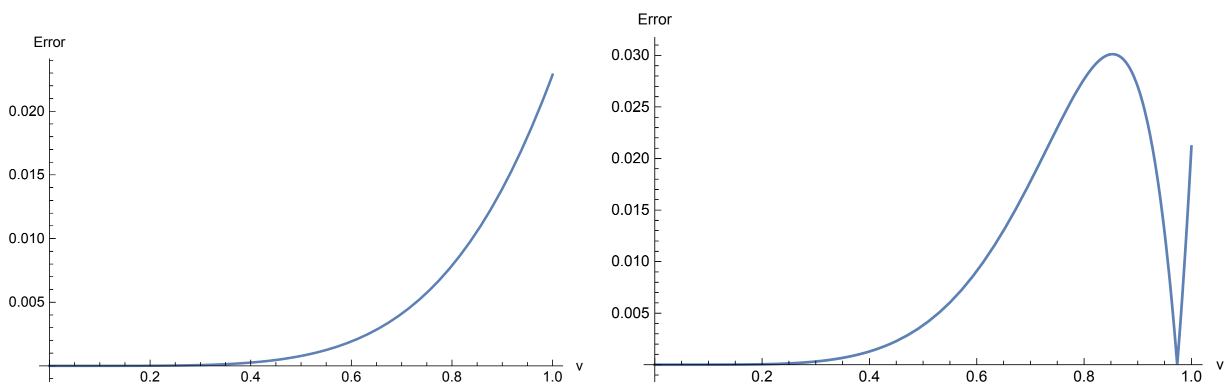

Figure 5: The absolute error for $(\underline{F}_{20}(v), \overline{F}_{20}(v))$ and $\mu = 0.5$.

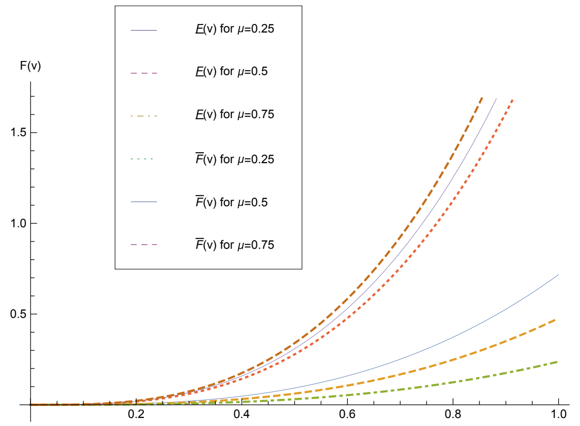

Figure 6: Fuzzy approximate solution for various μ.

6 Conclusion

In this work, the fuzzy Volterra integral equation of the second kind with piecewise kernel was studied.We applied the successive approximation scheme. This is the first time that the method has been implemented for solving this problem. The existence of an unique solution with the error bound and also the error estimation theorems were discussed. Some examples have been solved. Plotting the graphs of fuzzy approximate solutions for various μ and error functions we showed the accuracy of the method. Also the method has been compared with the traditional homotopy analysis method and we can see that the method is more accurate than the HAM. As the limitations of the method, generally the iterative methods are not fast thus when we need to make more iterations we need more time. Also for solving nonlinear problems if we have special and complicated nonlinear terms, applying the successive approximation method will not be easy. As our future works, we will combine the method with the CESTAC-CADNA strategy to find the numerical optimality results and optimal distance.

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